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Asymptotics and functional form of correlators in the XX-spin chain of finite length

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Abstract
We verify the functional form of the asymptotics of the spin–spin equal-time correlation function for the XX chain, predicted by the hypothesis of conformal invariance at large distances and by the bosonization procedure. We point out that the bosonization procedure also predicts the functional form of the correlators for the chains of finite length. We found the exact expression for the spin–spin equal-time correlator on a finite lattice. We find excellent agreement of the exact correlator with the prediction given by the leading asymptotics result up to very small distances. We also establish the correspondence between the value of the constant before the asymptotics for the XX chain with the expression for this constant proposed by Lukyanov and Zamolodchikov. We also evaluate the constant corresponding to the subleading term in the asymptotics in a way which is different from previous studies.

1. Introduction

It is well known that the hypothesis of conformal invariance at large distances predicts both the critical exponents and the functional form of the correlation functions for massless one-dimensional systems on the circle of finite length L (for example, see [1]). In this approach the conformal mapping $w = \frac{L}{2\pi}\ln(z)$ of the infinite plane z on the strip of finite width $w = t + \mathrm{i}x$, $x \in (0, L)$, is employed. It is known that in the case of the periodic boundary conditions (for the initial spin operators), the low-energy theory is the conformal field theory (CFT) with central charge $c = 1$ (which can be computed for the XXZ chain for arbitrary boundary conditions), which is equivalent to the so-called Gaussian model of CFT with $c = 1$, with a known spectrum of primary operators. The only parameter is the compactification radius (see below). I would like to stress also that application of the CFT for the calculation of, for example, critical indices is the hypothesis in a sense that it is not proved in a mathematically rigorous way. Thus any independent verification of the CFT predictions in the exactly solvable models is of interest. Alternatively, one can use the bosonization procedure for the low-energy effective theory for the systems in the gapless regime (Luttinger liquid) in its different versions [2, 3] to predict the critical exponents for various one-dimensional systems (for the first calculation of the critical index for the XXZ spin chain see [4]). In our opinion it is not clear in the literature that the bosonization also predicts the functional form of the correlators in the form $\sim\cos(2p_F x)((1/L)\sin(\pi x/L))^{\alpha}$, where L is the length of the system, α is the critical index and p_F is the Fermi momentum at sufficiently large distances. So it is worth pointing out how this functional form appears in the bosonization framework. This is done in the first part of this paper. Next, it is interesting to verify this functional form in the model where the correlation functions can be calculated exactly. In the present paper we study the spin–spin equal-time correlator in the XX spin chain (the Hamiltonian is $H = \frac{1}{2}\sum_{i=1}^{L}(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y)$ and the anisotropy parameter is $\Delta = 0$). It is worth mentioning that, although the XX chain can be solved with the help of mapping to the free-fermion system via the Jordan–Wigner transformation, the system is essentially the hard-core bosons on the lattice (at half-filling) and is not equivalent to the free fermions which manifest, for example, in completely different correlators in the two models. We show in the present paper that the spin–spin correlator in the XX model can be calculated exactly on a chain of finite length at any distance x of the order of the chain length (see equation (9) below). We show that the functional form of the correlator given by (9) coincides with that given by the CFT (bosonization). Note also that knowledge of the functional form of the correlators can be useful in the numerical study of correlators and extracting the critical exponents in different models. In section 2 we show how the usual bosonization procedure leads to the functional form predicted by the conformal invariance. In section 3 we briefly review the well known calculation of the spin–spin correlator in the XX model [5, 6] and present the expression for the correlator at distances $x \sim L$. Finally in section 4 we compare the numerical constant in the asymptotic of the correlator given in [6] with the expression for this constant proposed recently in [7] and evaluate the coefficient corresponding to the subleading term in the correlator [6, 9] in a new way.

2. Bosonization

Consider the effective low-energy Hamiltonian which builds up from the fermionic operators ($a_k, c_k, k = 2\pi n/L, n \in Z$, where L is the length of the chain) corresponding to the excitations around the right and the left Fermi points and consists of the kinetic energy term and the interaction term $H = T + V$ with the coupling constant λ:

$$H = \sum_k k(a_k^+a_k - c_k^+c_k) + \lambda/L\sum_{k,k',q} a_k^+a_{k+q}c_{k'}^+c_{k'-q}. \tag{1}$$

Defining the operators [2]

$$\rho_1(p) = \sum_k a_{k+p}^+a_k, \qquad \rho_2(p) = \sum_k c_{k+p}^+c_k,$$

where $|k|, |k+p| < \Lambda$, where Λ is some cut-off energy, which for the states with the filled Dirac sea have the following commutational relations:

$$[\rho_1(-p);\rho_1(p')] = \frac{pL}{2\pi}\delta_{p,p'} \qquad [\rho_2(p);\rho_2(-p')] = \frac{pL}{2\pi}\delta_{p,p'},$$

one can represent the Hamiltonian in the following form:

$$H = \frac{2\pi}{L}\sum_{p>0}(\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)) + \lambda\sum_{p>0}\frac{2\pi}{L}(\rho_1(p)\rho_2(-p) + \rho_1(-p)\rho_2(p)).$$

First, for the case of free Dirac fermions $\lambda = 0$ the system is equivalent to the free scalar Bose field with the Lagrangian $L = \frac{1}{2}((\partial_t\phi)^2 - (\partial_x\phi)^2)$

$$\phi(x,t) = \frac{1}{\sqrt{L}}\sum_k \frac{1}{2\sqrt{|k|}}(b_k^+\mathrm{e}^{\mathrm{i}kx} + b_k\mathrm{e}^{-\mathrm{i}kx}),$$

where b_k, b_k^+ are Bose creation and annihilation operators $[b_{k'}; b_k^+] = \delta_{kk'}$ and $kx = |k|t - kx$. The free Hamiltonian takes the following form: $H = \sum_p |p| b_p^+ b_p$, where $p = 2\pi n/L$ and the equal-time commutation relations are $[\partial_t \phi(x); \phi(y)] = -\mathrm{i}\delta(x-y)$. The identification of these operators with the operators $\rho_{1,2}(p)$ defined above is

$$b_p^+(p>0) = \rho_1(p)\sqrt{2\pi/L|p|} \qquad b_p^+(p<0) = \rho_2(p)\sqrt{2\pi/L|p|}$$

for the free fermion case.

To evaluate the correlators in a system of finite length and make the connection with the CFT predictions, one can proceed as follows. First one can define the lattice fields $n_{1,2}(x)$ with the help of the Fourier transform as

$$\rho_{1,2}(p) = \sum_x \mathrm{e}^{\mathrm{i}px} n_{1,2}(x), \qquad n_{1,2}(x) = \frac{1}{L}\sum_p \mathrm{e}^{-\mathrm{i}px}\rho_{1,2}(p).$$

These fields have the physical meaning of the local number of fermions above the Fermi level at the right and the left Fermi points. In terms of these fields the Hamiltonian has the following form:

$$H = 2\pi \sum_x \left(\tfrac{1}{2}(n_1^2(x) + n_2^2(x)) + \lambda n_1(x) n_2(x)\right).$$

Considering the average distribution of the number of extra particles we obtain the ground state energy in the form

$$\Delta E = \frac{2\pi}{L}\left(\frac{1}{2}((\Delta N_1)^2 + (\Delta N_2)^2) + \lambda \Delta N_1 \Delta N_2\right),$$

where $\Delta N_{1,2}$ are the additional numbers of particles at the two Fermi points. One can also rewrite the ground state energy in the sector with the total number of particles and the momentum $\Delta N = \Delta N_1 + \Delta N_2$, $\Delta Q = \Delta N_1 - \Delta N_2$ in such a way that the total Hamiltonian takes the following form (this was first proposed in [8]):

$$H = u(\lambda)\sum_p |p| b_p^+ b_p + \frac{\pi}{2L} u(\lambda)[\xi(\Delta N)^2 + (1/\xi)(\Delta Q)^2], \tag{2}$$

where the parameters $u(\lambda) = (1-\lambda^2)^{1/2}$ and $\xi = ((1+\lambda)/(1-\lambda))^{1/2}$. Calculation of finite-size corrections to the energy of the ground state for the XXZ spin chain (see, for example, [10]) leads to the expression (2) and allows one to obtain the parameter ξ which leads to the predictions of critical indices according to the CFT. The calculation gives the value $\xi = 2(\pi - \eta)/\pi$, where the parameter η is connected with the anisotropy parameter of the XXZ chain as $\Delta = \cos(\eta)$. Next one establishes the commutational relations for the fields $n_{1,2}(x)$. We have

$$[n_1(x); n_1(y)] = \frac{1}{(2\pi)^2}\frac{2\pi}{L}\sum_p p\mathrm{e}^{\mathrm{i}p(x-y)}.$$

To take the continuum limit it is sufficient to extend the sum over the momentum to $p \in (\pm\infty)$ (note that initially it was implied as the sum in the limits $p \in (\pm\pi)$). Then we obtain

$$[n_1(x); n_1(y)] \to \frac{1}{(2\pi)^2}\int \mathrm{d}p\, p\mathrm{e}^{\mathrm{i}p(x-y)} = -\frac{\mathrm{i}}{2\pi}\delta'(x-y).$$

Then introducing the new variables

$$\tilde{n}_{1,2}(x) = \sqrt{2\pi}\, n_{1,2}(x),$$

we have the following commutational relations and the density of the Hamiltonian:

$$H = \tfrac{1}{2}(\tilde{n}_1(x)\tilde{n}_1(x) + \tilde{n}_2(x)\tilde{n}_2(x)) + \lambda \tilde{n}_1(x)\tilde{n}_2(x). \tag{3}$$

We also have the following commutational relations: $[\tilde{n}_1(x); \tilde{n}_1(y)] = \delta'(x-y)$. We then have the following conjugated field and the momenta:

$$\pi(x) = \frac{1}{\sqrt{2}}(\tilde{n}_1(x) - \tilde{n}_2(x)); \qquad \partial_x\phi(x) = \frac{1}{\sqrt{2}}(\tilde{n}_1(x) + \tilde{n}_2(x))$$

$$\phi(x) = \tilde{N}(x) = \tilde{N}_1(x) + \tilde{N}_2(x), \qquad \tilde{N}_{1,2}(x) = \int_0^x \mathrm{d}y\, \tilde{n}_{1,2}(y).$$

In terms of these variables the Hamiltonian takes the following form:

$$H = \tfrac{1}{2}\int_0^L \mathrm{d}x\, ((1-\lambda)\pi^2(x) + (1+\lambda)(\partial\phi(x))^2).$$

The Hamiltonian density can also be represented in the following form:

$$H = \tfrac{1}{2}u(\lambda)[(1/\xi)\pi^2(x) + \xi(\partial\phi(x))^2] = \tfrac{1}{2}u(\lambda)[\hat{\pi}^2(x) + (\partial\hat{\phi}(x))^2], \tag{4}$$

where

$$\pi(x) = \sqrt{\xi}\hat{\pi}(x), \qquad \phi(x) = (1/\sqrt{\xi})\hat{\phi}(x). \tag{5}$$

The last equation (5) is nothing other than the canonical transformation, which is equivalent to the Bogoliubov transformation for the original operators $\rho_{1,2}(p)$. Next, to establish the expressions for fermions one should use the commutational relations $[a^+(x); \rho_1(p)] = -\mathrm{e}^{\mathrm{i}px}a^+(x)$ and the same for $c^+(x)$. Note that these last relations were obtained using the expression with original lattice fermions: $\rho_1(p) = \sum_y \mathrm{e}^{\mathrm{i}py}a^+(y)a(y)$. In this way we obtain the following expressions for fermionic operators:

$$\begin{aligned} a^+(x) &= K_1 \exp\left(\frac{2\pi}{L}\sum_{p\neq 0}\frac{\rho_1(p)}{p}\mathrm{e}^{-\mathrm{i}px}\right) = K_1\exp(-\mathrm{i}2\pi N_1(x)), \\ c^+(x) &= K_2 \exp\left(-\frac{2\pi}{L}\sum_{p\neq 0}\frac{\rho_2(p)}{p}\mathrm{e}^{-\mathrm{i}px}\right) = K_2\exp(\mathrm{i}2\pi N_2(x)), \end{aligned} \tag{6}$$

where the fields $N_{1,2}(x)$ differ by the normalization $\sqrt{2\pi}$ from the fields $\tilde{N}_{1,2}(x)$ and $K_{1,2}$ are the Klein factors—the operators which create the single particle at the right (left) Fermi points (we omit here the usual exponential expression $\mathrm{e}^{-\alpha|p|/2}$ in the exponent and the constant factor in front of the exponent $1/\sqrt{2\pi\alpha}$ which, in the limit $\alpha \to 0$, leads to the correct anticommutational relations [4]). Note that the above expressions (6) are equivalent to the known 'field-theoretical' bosonization formulae [3]:

$$a^+(c^+)(x) = K_{1,2}\exp\left(-\mathrm{i}\sqrt{\pi}\left(\int_{-\infty}^x \mathrm{d}y\,\pi(y) \pm \phi(x)\right)\right).$$

Now let us apply the above formulae to the specific case of the XXZ spin chain. Using the Jordan–Wigner transformation $\sigma_x^+ = a_x^+\exp(\mathrm{i}\pi N(x))$, where a_x^+ stands for the 'original' lattice fermionic operator, and performing the obvious substitutions $N(x) \to x/2 + N_1(x) + N_2(x)$ and $a_x^+ \to \mathrm{e}^{\mathrm{i}p_Fx}a^+(x) + \mathrm{e}^{-\mathrm{i}p_Fx}c^+(x)$, $p_F = \pi/2$, we obtain after the canonical transformation (5) the expression for the spin operator which determines the leading term in asymptotics of the correlator for the XXZ chain:

$$\sigma_x^+ \sim (-1)^x\exp(-\mathrm{i}\pi\sqrt{\xi}(\hat{N}_1(x) - \hat{N}_2(x))), \tag{7}$$

where $\hat{N}_{1,2}(x)$ corresponds to the free fields $\hat{\pi}(x)$, $\hat{\phi}(x)$, obtained after the transformation (5). To these operators correspond the new operators $\rho_{1,2}(p)$ and the new fermionic operators (quasiparticles). Analogously the term responsible for the subleading asymptotics has the form

$$\exp(\mathrm{i}2\pi(1/\sqrt{\xi})(\hat{N}_1(x) + \hat{N}_2(x)))\exp(-\mathrm{i}\pi\sqrt{\xi}(\hat{N}_1(x) - \hat{N}_2(x))).$$

Averaging the product of exponents in bosonic operators for the expression (7) and using the properties of $\rho_{1,2}(p)$, $\langle\rho_1(-p)\rho_1(p)\rangle = \frac{pL}{2\pi}\theta(p)$ and $\langle\rho_2(p)\rho_2(-p)\rangle = \frac{pL}{2\pi}\theta(p)$, we get for the correlation function $G(x) = \langle 0|\sigma^+_{i+x}\sigma^-_i|0\rangle$ the following sum in the exponent:

$$C\exp\left(\frac{\xi}{4}\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{e}^{\mathrm{i}n(2\pi x/L)} + \mathrm{h.c.}\right),$$

where C is some constant. Then using the formula $\sum_{n=1}^{\infty}\frac{1}{n}z^n = -\ln(1-z)$ and substituting the value $\xi = 2(\pi-\eta)/\pi \to 1$ we obtain the following expression for the XX chain:

$$G(x) = C_0\frac{(-1)^x}{(L\sin(\frac{\pi x}{L}))^{\alpha}}, \qquad \alpha = \frac{\pi-\eta}{\pi} = 1/2, \qquad (x \gg 1). \tag{8}$$

Thus, although bosonization, which deals with the low-energy effective theory, is not able to predict the constant before the asymptotics, the critical exponent and the functional form are predicted in accordance with CFT.

3. Asymptotics of correlators in XX spin chain

Let us briefly review the exact calculation [5] of the spin–spin equal-time correlation function (density matrix) for the XX spin chain on a finite lattice of length L:

$$G(x) = \langle 0|\sigma^+_{i+x}\sigma^-_i|0\rangle.$$

Using the Jordan–Wigner transformation relating spin operators to the Fermi operators (a^+_i, a_i) $\sigma^+_x = \exp(\mathrm{i}\pi\sum_{l<x} n_l)a^+_x$, the correlation function $G(x)$ can be represented as the following average over the free-fermion ground state:

$$G(x) = \langle 0|a^+_x\mathrm{e}^{\mathrm{i}\pi N(x)}a_0|0\rangle,$$

where $N(x) = \sum_{i=1}^{x-1} n_i$. Introducing the operators, anticommuting at different sites,

$$A_i = a^+_i + a_i, \qquad B_i = a^+_i - a_i, \qquad A_iB_i = \mathrm{e}^{\mathrm{i}\pi n_i},$$

where $n_i = a^+_i a_i$ is the fermion occupation number, with the following correlators with respect to the free-fermion vacuum:

$$\langle 0|B_iA_j|0\rangle = 2G_0(i-j), \qquad \langle 0|A_iA_j|0\rangle = 0, \qquad \langle 0|B_iB_j|0\rangle = 0,$$

where the free-fermion Green function on a finite chain $G_0(x)$ is

$$G_0(x) = \langle 0|a^+_{i+x}a_i|0\rangle = \frac{\sin(\pi x/2)}{L\sin(\pi x/L)},$$

one obtains the following expression for the bosonic correlator:

$$G(x) = \tfrac{1}{2}\langle 0|B_0(A_1B_1)(A_2B_2)\dots(A_{x-1}B_{x-1})A_x|0\rangle.$$

Note that we assume the periodic boundary conditions for the initial spin operators so that, strictly speaking, the above formulae are valid only for the case when L is even (the ground state is not degenerate) and $M = L/2$ is an odd integer (so that $L/4$ is not an integer). In fact, it is easily seen that for the boundary term $\sim\exp(\mathrm{i}\pi(M-1)) = +1$ when M is odd, the momentums of fermions are integer (not half-integer), their configuration is symmetric around zero and the free-fermionic Green function $G_0(x)$ is given exactly by the above formula. It was also mentioned in [5] that the periodic boundary conditions for spin operators ('a-cyclic' in this paper) lead to the same results as for the periodic boundary conditions for the fermionic operators ('c-cyclic') for the case when $L/4$ is not an integer. Using Wick's theorem we obtain the following determinant of the $x \times x$ matrix:

$$G(x) = \det_{ij}(2G_0(i-j-1)), \qquad i, j = 1, \dots, x.$$

Due to the form of this matrix ($G_0(l) = 0$ for even l) this determinant can be simplified and the following formulae are obtained:

$$G(x) = \tfrac{1}{2}(R_N)^2, \qquad (x = 2N),$$
$$G(x) = -\tfrac{1}{2}R_N R_{N+1}, \qquad (x = 2N+1),$$

where we denote by R_N the following determinant of the $N \times N$ matrix:

$$R_N = \det_{ij}((-1)^{i-j} G_0(2i - 2j - 1)), \qquad i, j = 1, \ldots N,$$

where $G_0(x)$ is the same Green function of free fermions as above. Since R_N is the Cauchy determinant one can obtain the following expression for it on the finite chain:

$$R_N = \left(\frac{2}{\pi}\right)^N \prod_{k=1}^{N-1} \left(\frac{(\sin(\pi(2k)/L))^2}{\sin(\pi(2k+1)/L)\sin(\pi(2k-1)/L)} \right)^{N-k}. \tag{9}$$

This formula is the main result of the present paper. Note that we obtained the exact expression for the correlator on the finite lattice. As was mentioned above, expression (9) for the correlator was obtained for M odd. As for the case with M even, it is clear on general grounds that in this case the correlator $G(x)$ will be modified by the terms of order $1/L$ at any distance x. (This can be proved rigorously. However, the proof will not be considered in the present paper.) From the expression (9) it is easy to obtain the correlator in the thermodynamic limit ($L \to \infty$) which is given by the similar product. We consider the analytical evaluation of this product for large N in the next section. For the finite chain it is easy to evaluate (9) numerically and compare the result with the asymptotic (8) at $x \gg 1$ and $x \sim L$. One finds numerically that, already at the sufficiently small distances x, where the contribution of the subleading term $\sim x^{-5/2}$ becomes negligible, the correction to the asymptotic formula (8) behaves like $\sim 1/L$ (at the very large $L \sim 10^5$ it is difficult to evaluate the product at $x \sim L$ due to numerical reasons). We find numerically that the exact correlator coincides with the correlator given by (9) with very high accuracy up to very small distances $x \sim 1$. One can also consider the limit $L \to \infty$ in (9) and, using the exact value of $G(x)$ at $x \sim 1$, estimate the value of the subleading term which is predicted to be of order $\sim 1/x^{\alpha+1/\alpha}$ for the XXZ chain. The coefficient for this term appears to be very small: at $x \sim 1$ (see section 4, equation (13)) so that the correlator for the infinite chain can be seen in very good agreement with the asymptotic formula (8) even at very small distances $x \sim 1$. The contribution of the next terms is difficult to obtain numerically (presumably due to their smallness). The physical reason for the smallness of the subleading term is not clear at the moment.

4. Numerical constant

Here we calculate the constant C_0 in front of the asymptotics of the density matrix in the XX spin chain and derive the representation proposed in [7] for this case. Using the expressions in the thermodynamic limit we calculate the product:

$$R_N = \left(\frac{2}{\pi}\right)^N \prod_{k=1}^{N-1} \left(\frac{(2k)^2}{(2k+1)(2k-1)} \right)^{N-k}.$$

Considering the logarithm of R_N we obtain the following sum:

$$\ln(R_N) = N \ln(2/\pi) + \sum_{p=1}^{\infty} \frac{1}{p} \left(\frac{1}{2}\right)^{2p} \sum_{k=1}^{N-1} (N-k) \frac{1}{k^{2p}}.$$

Next, we use the following formula for the finite sum $\sum_{k=1}^{n} 1/k^s = \zeta(s) - \zeta(s, n+1)$ (where $\zeta(s, n)$ is a generalized Riemannian zeta function) [12]:

$$\sum_{k=1}^{n} \frac{1}{k^m} = \frac{(-1)^m}{(m-1)!}(\psi^{(m-1)}(1) - \psi^{(m-1)}(n+1)),$$

where $\psi^{(n)}(z)$ is the nth derivative of the ψ function $\psi(z) = \Gamma'(z)/\Gamma(z)$, which leads to the following expression for $\ln(R_N)$:

$$\begin{aligned}\ln(R_N) = &\sum_{p=1}^{\infty} \frac{1}{p}\left(\frac{1}{2}\right)^{2p} \frac{1}{(2p-2)!}\psi^{(2p-2)}(1) \\ &- N\sum_{p=1}^{\infty} \frac{1}{p}\left(\frac{1}{2}\right)^{2p} \frac{1}{(2p-1)!}\psi^{(2p-1)}(N) \\ &- \sum_{p=1}^{\infty} \frac{1}{p}\left(\frac{1}{2}\right)^{2p} \frac{1}{(2p-2)!}\psi^{(2p-2)}(N),\end{aligned} \tag{10}$$

since the sum

$$N\sum_{p=1}^{\infty} \frac{1}{p}\left(\frac{1}{2}\right)^{2p} \frac{1}{(2p-1)!}\psi^{(2p-1)}(1) = -N\ln(2/\pi)$$

cancels the term coming from the factor $(2/\pi)^N$. The last formula can be obtained using $\psi^{(n)}(1) = (-1)^{n+1} n!\zeta(n+1)$ and rewriting the last sum using the definition of a ζ function as $-N\sum_{k=1}^{\infty} \ln(1 - 1/4k^2) = -N\ln(2/\pi)$, which can be seen from the infinite product representation $\sin(x)/x = \prod_{k=1}^{\infty}(1 - \pi^2 x^2/k^2)$. Thus we are left with three terms in equation (10). To find their asymptotic behaviour at large N up to terms of order $\sim 1/N^2$, we use the asymptotics

$$\psi(z) = \ln(z) - \frac{1}{2z} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2nz^{2n}}, \qquad |z| \to \infty,$$

where B_{2n} are Bernoulli numbers ($B_2 = 1/6$). One can see that the sum of the second and third terms in equation (10) is

$$-\frac{1}{4}\ln N - \frac{1}{4} - \frac{1}{64\,N^2} + \mathrm{O}\left(\frac{1}{N^4}\right), \tag{11}$$

while the first term equals

$$-\frac{1}{4}\gamma + \sum_{p=2}^{\infty} \frac{1}{p}\left(\frac{1}{2}\right)^{2p} \frac{1}{(2p-2)!}\psi^{(2p-2)}(1),$$

where $\gamma = -\psi(1) = 0.5772\ldots$ is Euler's constant. To estimate this sum one can use the integral representation

$$\psi(z) = \int_0^{\infty} \mathrm{d}t\left(\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-zt}}{1 - \mathrm{e}^{-t}}\right).$$

One also readily obtains the integral representation for $\psi^{(2p-2)}(1)$, which leads to the following expression for the first term:

$$\frac{1}{4}\int_0^{\infty} \mathrm{d}t\left(\frac{\mathrm{e}^{-t}}{t} - \frac{\mathrm{e}^{-t}}{1 - \mathrm{e}^{-t}}\left(\sum_{p=1}^{\infty} \frac{1}{p}\frac{1}{(2p-2)!}\left(\frac{t}{2}\right)^{2p-2}\right)\right).$$

Thus, combining all terms after the simple algebra, using the integration by parts and taking into account the constant term $-1/4$ in equation (11), we finally obtain the result:

$$\ln(R_N) = -\frac{1}{4}\ln(N) + \frac{1}{4}\int_0^\infty \frac{dt}{t}\left(e^{-4t} - \frac{1}{(\mathrm{ch}(t))^2}\right) - \frac{1}{64N^2}, \tag{12}$$

where the omitted terms are of order $\sim 1/N^4$. One can see that the above expression coincides with the formula proposed in [7] in the case of the XX chain. The last term in (12) gives the following coefficient for the next-to-leading asymptotics for the correlator:

$$G(x) \simeq \frac{C_0}{\sqrt{\pi}}\left((-1)^x \frac{1}{x^{1/2}} - \frac{1}{8}\frac{1}{x^{5/2}}\right), \tag{13}$$

where the constant C_0 is defined in (8). The value of the constant corresponding to the subleading term in equation (13) was first obtained by McCoy [6] using the method from [9] (with the help of the asymptotics of the Barnes G function [11]). Note that, using the asymptotics of $\psi(z)$, one can find analytically the higher terms in the asymptotics of the correlator. One could also represent the first term in equation (10) as

$$-\frac{1}{2}\gamma + \sum_{p=2}^{\infty} \frac{1}{p}\left(\frac{1}{2}\right)^{2p} \zeta(2p-1),$$

which also allows a numerical estimation of the constant. One can evaluate the integral in (12) to get the asymptotic

$$\ln(R_N) = -\frac{1}{4}\ln(N) + \left(\frac{\ln 2}{12} + 3\zeta'(-1)\right),$$

which is equivalent to the estimate

$$R_N = (N^{-1/4})(2^{1/12}e^{1/4}A^{-3}),$$

where A is the Glaisher constant:

$$A = e^{1/12-\zeta'(-1)} = 1.282\,427\ldots.$$

This result agrees with the result obtained by Wu [9], using the expression of the product through the Barnes G functions [11]. Finally, an alternative way to obtain the integral in equation (12) is to use the relations for the Γ function $\Gamma(n+1) = n!$, $\Gamma(n+1/2) = \pi^{1/2}2^{-n}(2n-1)!!$ to represent R_N in the form

$$R_N = \prod_{k=1}^{N} \frac{(\Gamma(k))^2}{\Gamma(k+1/2)\Gamma(k-1/2)}.$$

Then one can use the known integral representation for the sum of the type $\sum_{k=1}^{N} \ln\Gamma(k+b)$ (see [12]), and extract the large N asymptotics of the resulting expression. Thus for the constant before the asymptotic (8) the value $C_0/2\sqrt{\pi} = 0.147\,088\ldots$ is obtained.

5. Conclusion

In conclusion, we verified the functional form of the asymptotics of the spin–spin equal-time correlation function for the XX chain predicted by different methods. We find excellent agreement of the exact correlator with the prediction given by the leading asymptotics result up to very small distances. We have also estimated the coefficient corresponding to the subleading correction [6, 9] in a different way and found the expression for the leading term in agreement with [7]. It is worth mentioning that the correlators for the XY spin chain and the Ising-like spin chain have been studied at non-zero temperature and in the time-dependent case in different limits in a number of papers (for example, see [13]). Let us mention that the exact correlator in the XX model (at least in the infinite L limit) can be obtained in the framework of the general approach based on the algebraic Bethe ansatz method (see for example [14]) as well as in the approach based on the form factors of the model [15] which is beyond the scope of the present paper. Let us mention also that studying the form factors with a small energy of the excited states can be the basis of the determination of the constant C_0. We postpone this question for a separate publication. It is possible that the form factor approach can be useful for the calculation of the constant C_0 in the general case of the XXZ spin model.

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